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On generalized stochastic fractional integrals and related inequalities

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Abstract The generalized mean-square fractional integrals $\mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}$ and $\mathcal{J}^{\sigma}_{\rho,\lambda,v-;\omega}$ of the stochastic process X are introduced. Then, for Jensen-convex and strongly convex stochastic proceses, the generalized fractional Hermite–Hadamard inequality is establish via generalized stochastic fractional integrals.

Keywords Hermite–Hadamard inequality, stochastic fractional integrals, convex stochastic process

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1 Introduction

In 1980, Nikodem [11] introduced convex stochastic processes and investigated their regularity properties. In 1992, Skwronski [17] obtained some further results on convex stochastic processes.

Let $(\Omega, \mathcal{A}, P)$ be an arbitrary probability space. A function $X : \Omega \to \mathbb{R}$ is called a random variable if it is $\mathcal{A}$-measurable. A function $X : I \times \Omega \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, is called a stochastic process if for every $t \in I$ the function $X(t, .)$ is a random variable.

Recall that the stochastic process $X : I \times \Omega \to \mathbb{R}$ is called

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(i) continuous in probability in interval I, if for all $t_0 \in I$ we have

$$P\text{-}\lim_{t \to t_0} X(t,.) = X(t_0,.),$$

where P-lim denotes the limit in probability.

(ii) *mean-square continuous* in the interval I, if for all $t_0 \in I$

$$\lim_{t \to t_0} E\left[\left(X(t) - X(t_0)\right)^2\right] = 0,$$

where $E[X(t)]$ denotes the expectation value of the random variable $X(t,.)$.

Obviously, *mean-square* continuity implies continuity in probability, but the converse implication is not true.

Definition 1. Suppose we are given a sequence $\{\Delta^m\}$ of partitions, $\Delta^m = \{a_{m,0}, \ldots, a_{m,n_m}\}$. We say that the sequence $\{\Delta^m\}$ is a normal sequence of partitions if the length of the greatest interval in the n-th partition tends to zero, i.e.,

$$\lim_{m \to \infty} \sup_{1 \le i \le n_m} |a_{m,i} - a_{m,i-1}| = 0.$$

Now we would like to recall the concept of the mean-square integral. For the definition and basic properties see [18].

Let $X : I \times \Omega \to \mathbb{R}$ be a stochastic process with $E[X(t)^2] < \infty$ for all $t \in I$. Let $[a,b] \subset I$, $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ be a partition of $[a,b]$ and $\Theta_k \in [t_{k-1}, t_k]$ for all $k = 1, \ldots, n$. A random variable $Y : \Omega \to \mathbb{R}$ is called the mean-square integral of the process X on $[a,b]$, if we have

$$\lim_{n \to \infty} E\left[\left(\sum_{k=1}^{n} X(\Theta_k)(t_k - t_{k-1}) - Y\right)^2\right] = 0$$

for all normal sequences of partitions of the interval $[a,b]$ and for all $\Theta_k \in [t_{k-1}, t_k]$, $k = 1, \ldots, n$. Then, we write

$$Y(\cdot) = \int_a^b X(s,\cdot)ds \text{ (a.e.)}.$$

For the existence of the mean-square integral it is enough to assume the mean-square continuity of the stochastic process X.

Throughout the paper we will frequently use the monotonicity of the mean-square integral. If $X(t,\cdot) \le Y(t,\cdot)$ (a.e.) in some interval $[a,b]$, then

$$\int_a^b X(t,\cdot)dt \le \int_a^b Y(t,\cdot)dt \text{ (a.e.)}.$$

Of course, this inequality is an immediate consequence of the definition of the mean-square integral.

Definition 2. We say that a stochastic processes $X : I \times \Omega \to \mathbb{R}$ is convex, if for all $\lambda \in [0, 1]$ and $u, v \in I$ the inequality

$$X\big(\lambda u + (1-\lambda)v, \cdot\big) \leq \lambda X(u, \cdot) + (1-\lambda)X(v, \cdot) \quad \text{(a.e.)} \tag{1}$$

is satisfied. If the above inequality is assumed only for $\lambda = \frac{1}{2}$, then the process X is Jensen-convex or $\frac{1}{2}$-convex. A stochastic process X is concave if $(-X)$ is convex. Some interesting properties of convex and Jensen-convex processes are presented in [11, 18].

Now, we present some results proved by Kotrys [6] about Hermite–Hadamard inequality for convex stochastic processes.

Lemma 1. *If $X : I \times \Omega \to \mathbb{R}$ is a stochastic process of the form $X(t, \cdot) = A(\cdot)t + B(\cdot)$, where $A, B : \Omega \to \mathbb{R}$ are random variables, such that $E[A^2] < \infty$, $E[B^2] < \infty$ and $[a, b] \subset I$, then*

$$\int_a^b X(t, \cdot)dt = A(\cdot)\frac{b^2 - a^2}{2} + B(\cdot)(b - a) \text{ (a.e.).}$$

Proposition 1. *Let $X : I \times \Omega \to \mathbb{R}$ be a convex stochastic process and $t_0 \in intI$. Then there exists a random variable $A : \Omega \to \mathbb{R}$ such that X is supported at t_0 by the process $A(\cdot)(t - t_0) + X(t_0, \cdot)$. That is*

$$X(t, \cdot) \geq A(\cdot)(t - t_0) + X(t_0, \cdot) \text{ (a.e.).}$$

for all $t \in I$.

Theorem 1. *Let $X : I \times \Omega \to \mathbb{R}$ be a Jensen-convex, mean-square continuous in the interval I stochastic process. Then for any $u, v \in I$ we have*

$$X\left(\frac{u+v}{2}, \cdot\right) \leq \frac{1}{v-u}\int_u^v X(t, \cdot)dt \leq \frac{X(u, \cdot) + X(v, \cdot)}{2} \text{ (a.e.)} \tag{2}$$

In [7], Kotrys introduced the concept of strongly convex stochastic processes and investigated their properties.

Definition 3. Let $C : \Omega \to \mathbb{R}$ denote a positive random variable. The stochastic process $X : I \times \Omega \to \mathbb{R}$ is called strongly convex with modulus $C(\cdot) > 0$, if for all $\lambda \in [0, 1]$ and $u, v \in I$ the inequality

$$X\big(\lambda u + (1-\lambda)v, \cdot\big) \leq \lambda X(u, \cdot) + (1-\lambda)X(v, \cdot) - C(\cdot)\lambda(1-\lambda)(u-v)^2 \qquad \text{a.e.}$$

is satisfied. If the above inequality is assumed only for $\lambda = \frac{1}{2}$, then the process X is strongly Jensen-convex with modulus $C(\cdot)$.

In [5], Hafiz gave the following definition of stochastic mean-square fractional integrals.

Definition 4. For the stochastic proces $X : I \times \Omega \to \mathbb{R}$, the concept of stochastic mean-square fractional integrals I^{α}_{u+} and I^{α}_{v+} of X of order $\alpha > 0$ is defined by

$$I^{\alpha}_{u+}[X](t) = \frac{1}{\Gamma(\alpha)} \int_{u}^{t} (t-s)^{\alpha-1} X(x,s)ds \quad (a.e.), \quad t > u,$$

and

$$I^{\alpha}_{v-}[X](t) = \frac{1}{\Gamma(\alpha)} \int_{t}^{v} (s-t)^{\alpha-1} X(x,s)ds \quad (a.e.), \quad t < v.$$

Using this concept of stochastic mean-square fractional integrals I^{α}_{a+} and I^{α}_{b+}, Agahi and Babakhani proved the following Hermite–Hadamard type inequality for convex stochastic processes:

Theorem 2. *Let $X : I \times \Omega \to \mathbb{R}$ be a Jensen-convex stochastic process that is mean-square continuous in the interval I. Then for any $u, v \in I$, the following Hermite–Hadamard inequality*

$$X\left(\frac{u+v}{2}, \cdot\right) \leq \frac{\Gamma(\alpha+1)}{2(v-u)^{\alpha}} \left[I^{\alpha}_{u+}[X](v) + I^{\alpha}_{v-}[X](u)\right] \leq \frac{X(u,\cdot)+X(v,\cdot)}{2} \ (a.e.) \tag{3}$$

holds, where $\alpha > 0$.

For more information and recent developments on Hermite–Hadamard type inequalities for stochastic process, please refer to [2–4, 9–11, 14, 16, 15, 20, 19].

2 Main results

In tis section, we introduce the concept of the generalized mean-square fractional integrals $\mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}$ and $\mathcal{J}^{\sigma}_{\rho,\lambda,v-;\omega}$ of the stochastic process X.

In [13], Raina studied a class of functions defined formally by

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(x) = \mathcal{F}^{\sigma(0),\sigma(1),\ldots}_{\rho,\lambda}(x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k+\lambda)} x^k \quad \left(\rho,\lambda > 0; |x| < \mathcal{R}\right), \tag{4}$$

where the cofficents $\sigma(k)$ $(k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ make a bounded sequence of positive real numbers and $\mathcal{R}$ is the set of real numbers. For more information on the function (4), please refer to [8, 12]. With the help of (4), we give the following definition.

Definition 5. Let $X : I \times \Omega \to \mathbb{R}$ be a stochastic process. The generalized mean-square fractional integrals $\mathcal{J}^{\alpha}_{\rho,\lambda,a+;\omega}$ and $\mathcal{J}^{\alpha}_{\rho,\lambda,b-;\omega}$ of X are defined by

$$\mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}[X](x) = \int_{u}^{x} (x-t)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(x-t)^{\rho}\right] X(t,\cdot)dt, \quad \text{(a.e.)} \ x > u, \tag{5}$$

and

$$\mathcal{J}^{\sigma}_{\rho,\lambda,v-;\omega}[X](x) = \int_{x}^{v} (t-x)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(t-x)^{\rho}\right] X(s,\cdot)dt, \quad \text{(a.e.)} \ x < v, \tag{6}$$

where $\lambda, \rho > 0$, $\omega \in \mathbb{R}$.

Many useful generalized mean-square fractional integrals can be obtained by specializing the coefficients $\sigma(k)$. Here, we just point out that the stochastic mean-square fractional integrals I^{α}_{a+} and I^{α}_{b+} can be established by coosing $\lambda = \alpha$, $\sigma(0) = 1$ and $w = 0$.

Now we present Hermite–Hadamard inequality for generalized mean-square fractional integrals $\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}$ and $\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}$ of X.

Theorem 3. *Let $X : I \times \Omega \to \mathbb{R}$ be a Jensen-convex stochastic process that is mean-square continuous in the interval I. For every $u, v \in I$, $u < v$, we have the following Hermite–Hadamard inequality*

$$\begin{aligned} X\left(\frac{u+v}{2}, \cdot\right) & \\ \le \frac{1}{2(v-u)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(v-u)^{\rho}]} & \left[\mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}[X](t) + \mathcal{J}^{\sigma}_{\rho,\lambda,v-;\omega}[X](t)\right] \\ \le \frac{X(u,\cdot)+X(v,\cdot)}{2}. & \qquad a.e. \end{aligned} \tag{7}$$

Proof. Since the process X is mean-square continuous, it is continuous in probability. Nikodem [11] proved that every Jensen-convex and continuous in probability stochastic process is convex. Since X is convex, then by Proposition 1, it has a supporting process at any point $t_0 \in intI$. Let us take a support at $t_0 = \frac{u+v}{2}$, then we have

$$X(t,\cdot) \ge A(\cdot)\left(t - \frac{u+v}{2}\right) + X\left(\frac{u+v}{2}, \cdot\right). \quad \text{a.e.} \tag{8}$$

Multiplying both sides of (8) by $[(v-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(v-t)^{\rho}] + (t-u)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}[\omega(t-u)^{\rho}]]$, then integrating the resulting inequality with respect to t over $[u, v]$, we obtain

$$\begin{aligned} &\int_u^v \left[(v-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(v-t)^{\rho}\right] + (t-u)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(t-u)^{\rho}\right]\right]X(t,\cdot)dt \\ &\quad \ge A(\cdot)\int_u^v \left[(v-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(v-t)^{\rho}\right]\right. \\ &\qquad \left. + (t-u)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(t-u)^{\rho}\right]\right]\left(t - \frac{u+v}{2}\right)dt \\ &\qquad + X\left(\frac{u+v}{2}, \cdot\right)\int_u^v \left[(v-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(v-t)^{\rho}\right]\right. \\ &\qquad \left. + (t-u)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(t-u)^{\rho}\right]\right]dt \\ &\quad = A(\cdot)\int_u^v \left[t(v-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(v-t)^{\rho}\right] + t(t-u)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(t-u)^{\rho}\right]\right]dt \\ &\qquad - A(\cdot)\frac{u+v}{2}\int_u^v \left[(v-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(v-t)^{\rho}\right] + (t-u)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(t-u)^{\rho}\right]\right]dt \end{aligned} \tag{9}$$

$$+ X\left(\frac{u+v}{2},\cdot\right)\int_u^v \left[(v-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(v-t)^{\rho}\right]\right.$$
$$\left. + (t-u)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(t-u)^{\rho}\right]\right]dt.$$

Calculating the integrals, we have

$$\begin{aligned}
&\int_u^v t(v-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(v-t)^{\rho}\right]dt \\
&\quad = -\int_u^v (v-t)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(v-t)^{\rho}\right]dt + v\int_u^v (v-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(v-t)^{\rho}\right]dt \\
&\quad = -(v-u)^{\lambda+1}\mathcal{F}^{\sigma_1}_{\rho,\lambda}\left[\omega(v-u)^{\rho}\right] + v(v-u)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(v-u)^{\rho}\right]
\end{aligned} \tag{10}$$

and similarly,

$$\begin{aligned}
&\int_u^v t(t-u)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(t-u)^{\rho}\right]dt \\
&\quad = \int_u^v (t-u)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(t-u)^{\rho}\right]dt + u\int_u^v (t-u)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(t-u)^{\rho}\right]dt \\
&\quad = (v-u)^{\lambda+1}\mathcal{F}^{\sigma_1}_{\rho,\lambda}\left[\omega(v-u)^{\rho}\right] + u(v-u)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(v-u)^{\rho}\right]
\end{aligned} \tag{11}$$

where $\sigma_1(k) = \frac{\sigma(k)}{\rho k+\lambda+1}$, $k = 0, 1, 2, \ldots$. Using the identities (10) and (11) in (9), we obtain

$$\begin{aligned}
&\mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}[X](t) + \mathcal{J}^{\sigma}_{\rho,\lambda,v-;\omega}[X](t) \\
&\quad \geq A(\cdot)(u+v)(v-u)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(v-u)^{\rho}\right] \\
&\qquad - A(\cdot)\frac{u+v}{2}2(v-u)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(v-u)^{\rho}\right] \\
&\qquad + X\left(\frac{u+v}{2},\cdot\right)2(v-u)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(v-u)^{\rho}\right] \\
&\quad = X\left(\frac{u+v}{2},\cdot\right)2(v-u)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(v-u)^{\rho}\right].
\end{aligned}$$

That is,

$$\begin{aligned}
&X\left(\frac{u+v}{2},\cdot\right) \\
&\quad \leq \frac{1}{2(v-u)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(v-u)^{\rho}]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}[X](t) + \mathcal{J}^{\sigma}_{\rho,\lambda,v-;\omega}[X](t)\right] \text{ a.e.,}
\end{aligned}$$

which completes the proof of the first inequality in (7).

By using the convexity of X, we get

$$X(t,\cdot) = X\left(\frac{v-t}{v-u}u + \frac{t-u}{v-u}v,\cdot\right) \le \frac{v-t}{v-u}X(u,\cdot) + \frac{t-u}{v-u}X(v,\cdot)$$
$$= \frac{X(v,\cdot)-X(u,\cdot)}{v-u}t + \frac{X(u,\cdot)v - X(v,\cdot)u}{v-u} \quad \text{a.e.}$$

for $t \in [u,v]$. Using the identities (10) and (11), it follows that

$$\int_u^v \left[(v-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(v-t)^{\rho}\right] + (t-u)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(t-u)^{\rho}\right]\right]X(t,\cdot)dt$$
$$\le \frac{X(v,\cdot)-X(u,\cdot)}{v-u}$$
$$\times \int_u^v \left[t(v-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(v-t)^{\rho}\right] + t(t-u)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(t-u)^{\rho}\right]\right]dt$$
$$+ \frac{X(u,\cdot)v - X(v,\cdot)u}{v-u}$$
$$\times \int_u^v \left[(v-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(v-t)^{\rho}\right] + (t-u)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(t-u)^{\rho}\right]\right]dt$$
$$= \frac{X(v,\cdot)-X(u,\cdot)}{v-u}(u+v)(v-u)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(v-u)^{\rho}\right]$$
$$+ \frac{X(u,\cdot)v - X(v,\cdot)u}{v-u}2(v-u)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(v-u)^{\rho}\right]$$
$$= \left[X(u,\cdot)+X(v,\cdot)\right](v-u)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(v-u)^{\rho}\right].$$

That is,

$$\frac{1}{2(v-u)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(v-u)^{\rho}]}\left[\mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}[X](t) + \mathcal{J}^{\sigma}_{\rho,\lambda,v-;\omega}[X](t)\right]$$
$$\le \frac{X(u,\cdot)+X(v,\cdot)}{2} \quad \text{a.e.,}$$

which completes the proof. □

Remark 1. *i) Choosing $\lambda = \alpha$, $\sigma(0) = 1$ and $w = 0$ in Theorem 3, the inequality (7) reduces to the inequality (3).*

ii) Choosing $\lambda = 1$, $\sigma(0) = 1$ and $w = 0$ in Theorem 3, the inequality (7) reduces to the inequality (2).

Theorem 4. *Let $X : I \times \Omega \to \mathbb{R}$ be a stochastic process, which is strongly Jensen-convex with modulus $C(\cdot)$ and mean-square continuous in the interval I so that $E[C^2] < \infty$. Then for any $u, v \in I$, we have*

$$X\left(\frac{u+v}{2},\cdot\right)$$

$$- C(\cdot)\Big\{2(v-u)^{\lambda+2}\mathcal{F}^{\sigma_2}_{\rho,\lambda}\big[\omega(v-u)^{\rho}\big] - 2(v-u)^{\lambda}\mathcal{F}^{\sigma_1}_{\rho,\lambda}\big[\omega(v-u)^{\rho}\big]$$
$$+ \big(u^2+v^2\big)(v-u)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\big[\omega(v-u)^{\rho}\big] - \left(\frac{u+v}{2}\right)^2\Big\}$$
$$\le \frac{1}{2(v-u)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(v-u)^{\rho}]}\big[\mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}[X](t) + \mathcal{J}^{\sigma}_{\rho,\lambda,v-;\omega}[X](t)\big]$$
$$\le \frac{X(u,\cdot)+X(v,\cdot)}{2} - C(\cdot)\Big\{\frac{u^2+v^2}{2} + 2(v-u)^{\lambda+2}\mathcal{F}^{\sigma_2}_{\rho,\lambda}\big[\omega(v-u)^{\rho}\big]$$
$$- 2(v-u)^{\lambda}\mathcal{F}^{\sigma_1}_{\rho,\lambda}\big[\omega(v-u)^{\rho}\big] + \big(u^2+v^2\big)(v-u)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\big[\omega(v-u)^{\rho}\big]\Big\} \text{ a.e.}$$

Proof. It is known that if X is strongly convex process with the modulus $C(\cdot)$, then the process $Y(t,\cdot) = X(t,\cdot) - C(\cdot)t^2$ is convex [7, Lemma 2]. Appying the inequality (7) for the process $Y(t,.)$, we have

$$Y\left(\frac{u+v}{2},\cdot\right) \le \frac{1}{2(v-u)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(v-u)^{\rho}]}\int_u^v \big[(v-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\big[\omega(v-t)^{\rho}\big]$$
$$+ (t-u)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\big[\omega(t-u)^{\rho}\big]\big]Y(t,\cdot)dt$$
$$\le \frac{Y(u,\cdot)+Y(v,\cdot)}{2} \qquad \text{a.e.}$$

That is

$$X\left(\frac{u+v}{2},\cdot\right) - C(\cdot)\left(\frac{u+v}{2}\right)^2$$
$$\le \frac{1}{2(v-u)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(v-u)^{\rho}]}\Big\{\int_u^v \big[(v-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\big[\omega(v-t)^{\rho}\big]$$
$$+ (t-u)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\big[\omega(t-u)^{\rho}\big]\big]X(t,.)dt$$
$$- C(\cdot)\int_u^v \big[t^2(v-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\big[\omega(v-t)^{\rho}\big] + t^2(t-u)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\big[\omega(t-u)^{\rho}\big]\big]dt\Big\}$$
$$\le \frac{X(u,\cdot) - C(\cdot)u^2 + X(v,\cdot) - C(\cdot)v^2}{2} \qquad \text{a.e.}$$

Calculating the integrals, we obtain

$$\int_u^v t^2(v-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\big[\omega(v-t)^{\rho}\big]dt$$
$$= \int_u^v t^2(v-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\big[\omega(v-t)^{\rho}\big]dt + \int_u^v t^2(v-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\big[\omega(v-t)^{\rho}\big]dt$$

$$+\int_u^v t^2(v-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\big[\omega(v-t)^{\rho}\big]dt$$
$$=(v-u)^{\lambda+2}\mathcal{F}^{\sigma_2}_{\rho,\lambda}\big[\omega(v-u)^{\rho}\big]-2v(v-u)^{\lambda+1}\mathcal{F}^{\sigma_1}_{\rho,\lambda}\big[\omega(v-u)^{\rho}\big]$$
$$+v^2(v-u)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\big[\omega(v-u)^{\rho}\big]$$

and similarly,

$$\int_u^v t^2(t-u)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\big[\omega(t-u)^{\rho}\big]dt$$
$$=(v-u)^{\lambda+2}\mathcal{F}^{\sigma_2}_{\rho,\lambda}\big[\omega(v-u)^{\rho}\big]+2u(v-u)^{\lambda+1}\mathcal{F}^{\sigma_1}_{\rho,\lambda}\big[\omega(v-u)^{\rho}\big]$$
$$+u^2(v-u)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\big[\omega(v-u)^{\rho}\big],$$

where $\sigma_2(k)=\frac{\sigma(k)}{\rho k+\lambda+2}$, $k=0,1,2,\ldots$. Then it follows that

$$X\left(\frac{u+v}{2},\cdot\right)-C(\cdot)\left(\frac{u+v}{2}\right)^2$$
$$\leq\frac{1}{2(v-u)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(v-u)^{\rho}]}\big[\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}[X](t)+\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}[X](t)\big]$$
$$-C(\cdot)\big[2(v-u)^{\lambda+2}\mathcal{F}^{\sigma_2}_{\rho,\lambda}\big[\omega(v-u)^{\rho}\big]$$
$$-2(v-u)^{\lambda}\mathcal{F}^{\sigma_1}_{\rho,\lambda}\big[\omega(v-u)^{\rho}\big]+\big(u^2+v^2\big)(v-u)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\big[\omega(v-u)^{\rho}\big]\big]$$
$$\leq\frac{X(u,\cdot)+X(v,\cdot)}{2}-C(\cdot)\frac{u^2+v^2}{2}\qquad\text{a.e.}$$

Then

$$X\left(\frac{u+v}{2},\cdot\right)$$
$$-C(\cdot)\Big\{2(v-u)^{\lambda+2}\mathcal{F}^{\sigma_2}_{\rho,\lambda}\big[\omega(v-u)^{\rho}\big]-2(v-u)^{\lambda}\mathcal{F}^{\sigma_1}_{\rho,\lambda}\big[\omega(v-u)^{\rho}\big]$$
$$+\big(u^2+v^2\big)(v-u)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\big[\omega(v-u)^{\rho}\big]-\left(\frac{u+v}{2}\right)^2\Big\}$$
$$\leq\frac{1}{2(v-u)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[\omega(v-u)^{\rho}]}\big[\mathcal{J}^{\sigma}_{\rho,\lambda,u+;\omega}[X](t)+\mathcal{J}^{\sigma}_{\rho,\lambda,v-;\omega}[X](t)\big]$$
$$\leq\frac{X(u,\cdot)+X(v,\cdot)}{2}-C(\cdot)\Big\{\frac{u^2+v^2}{2}+2(v-u)^{\lambda+2}\mathcal{F}^{\sigma_2}_{\rho,\lambda}\big[\omega(v-u)^{\rho}\big]$$
$$-2(v-u)^{\lambda}\mathcal{F}^{\sigma_1}_{\rho,\lambda}\big[\omega(v-u)^{\rho}\big]+\big(u^2+v^2\big)(v-u)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\big[\omega(v-u)^{\rho}\big]\Big\}\ \text{a.e.}$$

This completes the proof. □

Remark 2. *Choosing $\lambda=\alpha$, $\sigma(0)=1$ and $w=0$ in Theorem 4, it reduces to Theorem 7 in [1].*

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