

Asymptotic arbitrage in fractional mixed markets

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Abstract We consider a family of mixed processes given as the sum of a fractional Brownian motion with Hurst parameter $H \in (3/4, 1)$ and a multiple of an independent standard Brownian motion, the family being indexed by the scaling factor in front of the Brownian motion. We analyze the underlying markets with methods from large financial markets. More precisely, we show the existence of a strong asymptotic arbitrage (defined as in Kabanov and Kramkov [Finance Stoch. 2(2), 143–172 (1998)]) when the scaling factor converges to zero. We apply a result of Kabanov and Kramkov [Finance Stoch. 2(2), 143–172 (1998)] that characterizes the notion of strong asymptotic arbitrage in terms of the entire asymptotic separation of two sequences of probability measures. The main part of the paper consists of proving the entire separation and is based on a dichotomy result for sequences of Gaussian measures and the concept of relative entropy.

Keywords Mixed fractional Brownian motion, relative entropy, large financial market, entire asymptotic separation, strong asymptotic arbitrage

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1 Introduction

Empirical studies of financial time series led to the conclusion that the log-return increments exhibit long-range dependence. This fact supports the idea of modelling the randomness of a risky asset using a fractional Brownian motion with Hurst parameter $H > 1/2$. However, markets driven by a fractional Brownian motion have been extensively disputed, as this motion fails to be a semimartingale and, hence, they allow for a free lunch with vanishing risk (see [6]).

Many attempts were proposed to overcome this drawback of the fractional Brownian motion. In this work, we deal with the regularization method proposed by Cheridito in [2, 1] when $H > 3/4$. This method consists in adding to the fractional Brownian motion a multiple of an independent Brownian motion, the resulting process, called *mixed fractional Brownian motion*, being Gaussian with the long-range dependence property. Moreover, as shown in [2, 1], when $H > 3/4$ the mixed fractional Brownian motion is equivalent to a multiple of a Brownian motion. Therefore, a Black–Scholes type model in which the randomness of the risky asset is driven by a mixed fractional Brownian motion is arbitrage free and complete. We call such a model a *mixed fractional Black–Scholes model*.

On one hand the fractional Black–Scholes model admits arbitrage. On the other hand, when the Hurst parameter $H > 3/4$, adding a Brownian component (in the above explained way) makes the arbitrage disappear. In this paper we aim to go a step further and study the sensitivity to arbitrage of the mixed fractional Black–Scholes model when the Brownian component asymptotically vanishes. In [3, 4] it was argued that a good way of seeing the sensitivity to arbitrage of a market when one of its parameters converges to zero (or infinity), is to consider the family of markets indexed by the corresponding parameter and to use methods from large financial markets. To be precise, we study the asymptotic arbitrage opportunities in the sequence of mixed fractional Black–Scholes models when the scaling factor in front of the Brownian motion converges to zero. We focus on the notion of *strong asymptotic arbitrage* (SAA) introduced by Kabanov and Kramkov in [9] as the possibility of getting arbitrarily rich with probability arbitrarily close to one by taking a vanishing risk. Our model fits the standard framework of large financial markets, as each mixed fractional Black–Scholes model is arbitrage free (and even complete). We point out that the existence of arbitrage in the limiting market does not directly imply the existence of any kind of asymptotic arbitrage in the approximating sequence of mixed markets. In [9] the existence of strong asymptotic arbitrage was shown to be equivalent to the entire asymptotic separation of the sequence of objective probability measures and the sequence of equivalent martingale measures.

In order to show the existence of strong asymptotic arbitrage in the sequence of mixed fractional Black–Scholes models we use the result of [9] that was mentioned above. That means we show that the sequence of objective probability measures is entirely asymptotically separable from the sequence of equivalent martingale measures. Our main contribution is the proof of this entire asymptotic separability in the given model. We use the notion of relative entropy and a dichotomy result for sequences of Gaussian measures. Indeed, inspired by the work of Cheridito [2, 1], we first show, for each fixed market, that the entropy of the objective probability measure relative to

the equivalent martingale measure, both restricted to a discrete partition, converges to infinity. Our proof then follows using tightness arguments for the sequence of Radon–Nikodym derivatives of the objective probability measures with respect to equivalent martingale measures and the fact that two sequences of Gaussian measures are either mutually contiguous or entirely separable. The latter is known in the literature as the equivalence/singularity dichotomy for sequences of Gaussian processes, see [5].

The paper is structured as follows. In Section 2, we set the mixed fractional Black–Scholes model and recall the framework of the large financial market. At the end of this part, we state the main result (Theorem 1). Section 3 is dedicated to the proof of Theorem 1, whereas Section 4 provides a discussion about the existence of strong asymptotic arbitrage using self-financing strategies constrained to jump only in a finite set of times. We end our work with Appendix A in which we recall the definition of relative entropy and an equivalent characterization in terms of the Radon–Nikodym derivative.

2 Preliminaries and main results

2.1 Setting the model

Let (Ω, \mathcal{F}, P) be a probability space.

Definition 1. A fractional Brownian motion $Z^H = (Z_t^H)_{t \geq 0}$ with Hurst parameter $H \in (0, 1)$ is a continuous centred Gaussian process with covariance function

$$\text{Cov}(Z_t^H, Z_s^H) = E(Z_t^H Z_s^H) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}), \quad s, t \geq 0.$$

In particular, $Z^{\frac{1}{2}}$ is a standard Brownian motion.

A linear combination of different fractional Brownian motions is referred in the literature as a *mixed fractional Brownian motion*. In order to avoid localization arguments we only consider finite time horizon processes. In addition we focus on linear combinations of a standard Brownian motion $(B_t)_{t \in [0, 1]}$ and an independent fractional Brownian motion $(Z_t^H)_{t \in [0, 1]}$ with Hurst parameter $H \in (3/4, 1)$, both defined on (Ω, \mathcal{F}, P) . Cheridito shows in [2, 1] that, for each $\alpha \in \mathbb{R}$ the mixed process $M^{H, \alpha} := (M_t^{H, \alpha})_{t \in [0, 1]}$ defined by

$$M_t^{H, \alpha} := \alpha Z_t^H + B_t, \quad t \in [0, 1],$$

is equivalent to a Brownian motion. By this we mean that the measure $Q^{H, \alpha}$ induced on $\mathcal{C}[0, 1]$ by $M^{H, \alpha}$ and the Wiener measure Q_W (induced by the Brownian motion on $\mathcal{C}[0, 1]$) are equivalent. As a consequence, the process $M^{H, \alpha}$ is a $(\mathcal{F}_t^{H, \alpha})_{t \in [0, 1]}$ -semimartingale, where, for each $t \geq 0$, $\mathcal{F}_t^{H, \alpha} := \sigma((M_s^{H, \alpha})_{s \in [0, t]})$ is the right-continuous natural filtration augmented by the nullsets.

Now, for each $\alpha > 0$, we call by the α -mixed fractional Black–Scholes model the financial market consisting of a risk free asset normalized to one and a risky asset $(S_t^{H, \alpha})_{t \in [0, 1]}$ given by

$$S_t^{H, \alpha} := S_0^{H, \alpha} \exp\left(\left(\mu - \frac{\sigma^2}{2\alpha^2}\right)t + \sigma\left(Z_t^H + \frac{1}{\alpha}B_t\right)\right), \quad t \in [0, 1], \quad (1)$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$ represent the drift and the volatility of the asset.¹ We denote by $X := (X_t)_{t \in [0,1]}$ the coordinate process in $\mathcal{C}[0, 1]$ and we define the process $S^\alpha := (S_t^\alpha)_{t \in [0,1]}$ as

$$S_t^\alpha := S_0^\alpha \exp\left(\left(\mu - \frac{\sigma^2}{2\alpha^2}\right)t + \frac{\sigma}{\alpha}X_t\right), \quad t \in [0, 1]. \quad (2)$$

From the above discussion, we conclude that $S^{H,\alpha}$ under P is equivalent to S^α under Q_W , which is a martingale when $\mu = 0$. For a general drift, we denote by $Q_{\frac{\mu\alpha}{\sigma}}$ the measure induced on $\mathcal{C}[0, 1]$ by the Brownian motion with drift $-\frac{\mu\alpha}{\sigma}$ (in particular $Q_0 = Q_W$). Thanks to the Girsanov theorem, the process $S^{H,\alpha}$ under P is also equivalent to S^α under $Q_{\frac{\mu\alpha}{\sigma}}$, which is a martingale. Therefore, the α -mixed fractional Black–Scholes model with the filtration $(\mathcal{F}_t^{H,\alpha})_{t \in [0,1]}$ has a unique equivalent martingale measure, and therefore is arbitrage-free and complete.

2.2 Asymptotic arbitrage

In this work, we treat the collection of α -mixed fractional Black–Scholes models with methods from large financial markets. This idea is formalized in the following definition.

Definition 2 (The large mixed fractional market). We call by *large mixed fractional market* the family of α -mixed fractional Black–Scholes models, $\alpha > 0$, i.e. the family of markets

$$\mathbb{L}^H := (\Omega, \mathcal{F}, (\mathcal{F}_t^{H,\alpha})_{t \in [0,1]}, P, S^{H,\alpha})_{\alpha > 0}.$$

We aim to study the presence of asymptotic arbitrage in the large financial mixed fractional market when α tends to infinity, i.e. when the Brownian component asymptotically disappears. More precisely, we intend to investigate, using methods of [9], the presence of a so-called *strong asymptotic arbitrage*. The latter is an analogue concept of arbitrage but for sequences of markets rather than for a single market model. Intuitively, this kind of arbitrage for sequences of markets gives the possibility of getting arbitrarily rich with probability arbitrarily close to one while taking a vanishing risk. In order to make this idea precise we first specify the set of admissible trading strategies.

Definition 3 (Admissible trading strategy). A *trading strategy* for $S^{H,\alpha}$ is a real-valued predictable $S^{H,\alpha}$ -integrable stochastic process $\Phi := (\Phi_t)_{t \in [0,1]}$. The trading strategy is said to be *admissible* if there is $m \in \mathbb{R}_+$ such that for all $t \in [0, 1]$: $(\Phi \cdot S^{H,\alpha})_t \geq -m$ almost surely.

Now we proceed to recall the definition of strong asymptotic arbitrage of [9].

Definition 4. A strong asymptotic arbitrage (SAA) is said to exist in the large mixed fractional market as α tends to infinity if there exists a sequence $(\alpha_\ell)_{\ell \geq 1}$ converging to infinity and a sequence $(\Phi_\ell)_{\ell \geq 1}$, where Φ_ℓ is an admissible trading strategy for S^{H,α_ℓ} , such that

¹ $S^{H,\alpha}$ is the solution of $dS_t^{H,\alpha} = \mu S_t^{H,\alpha} dt + \sigma S_t^{H,\alpha} d(Z^H + \frac{1}{\alpha}B)_t$.

1. $(\Phi_\ell \cdot S^{H,\alpha_\ell})_t \geq -m_\ell, \quad t \in [0, 1], \ell \geq 1,$
2. $\lim_{\ell \rightarrow \infty} P((\Phi_\ell \cdot S^{H,\alpha_\ell})_1 \geq M_\ell) = 1,$

where m_k and M_k are sequences of positive real numbers converging to zero and to infinity, respectively.

Remark 1. This definition is equivalent to the notion of strong asymptotic arbitrage of the first kind as given in [9]. This is trivially seen by taking $V_0^\ell(\Phi_\ell) = \frac{m_\ell}{M_\ell}$ and $V_t^\ell(\Phi_\ell) = \frac{m_\ell}{M_\ell} + \frac{1}{M_\ell}(\Phi_\ell \cdot S^{H,\alpha_\ell})_t$ that the SAA1 of [9] can be obtained from our SAA. It is equally trivial to get a SAA from the SAA1. The notion is further equivalent to the strong asymptotic arbitrage of the second kind from [9] as it is shown there that SAA1 and SAA2 are equivalent and hence can be subsumed under the name SAA.

Our approach to show the existence of arbitrage of this kind will be not constructive. Instead, we use an equivalent characterization of strong asymptotic arbitrage based on the notion of *entire asymptotic separability of sequences of measures*, which is defined as follows.

Definition 5. The sequences of probability measures $(P_\ell)_{\ell \geq 1}$ and $(Q_\ell)_{\ell \geq 1}$ are said to be entirely asymptotically separable if there exists a subsequence ℓ_k and a sequence of sets $A_k \in \mathcal{F}^{\ell_k}$ such that $\lim_{k \rightarrow \infty} P_{\ell_k}(A_k) = 1$ and $\lim_{k \rightarrow \infty} Q_{\ell_k}(A_k) = 0$. In this case we write $(P_\ell)_{\ell \geq 1} \Delta (Q_\ell)_{\ell \geq 1}$. In addition, two families of probability measures $(P^\alpha)_{\alpha > 0}$ and $(Q^\alpha)_{\alpha > 0}$ are said to be entirely asymptotically separable, and we write $(P^\alpha)_{\alpha > 0} \Delta (Q^\alpha)_{\alpha > 0}$, if there is a sequence $(\alpha_\ell)_{\ell \geq 1}$ converging to infinity such that $(P^{\alpha_\ell})_{\ell \geq 1} \Delta (Q^{\alpha_\ell})_{\ell \geq 1}$.

The precise relation between this notion and the existence of SAA is given in [9, Proposition 4]. In the case of complete markets, this result takes the following simple form.

Proposition 1. Consider a large financial market $(\Omega^\alpha, \mathcal{F}^\alpha, (\mathcal{F}_t^\alpha)_{t \in [0, T]}, P^\alpha)_{\alpha > 0}$ and assume that each small market is complete. For each $\alpha > 0$, let $Q^\alpha \sim P^\alpha$ be the unique equivalent martingale measure. Then the following conditions are equivalent

1. There is a SAA.
2. $(P^\alpha)_{\alpha > 0} \Delta (Q^\alpha)_{\alpha > 0}$.

Therefore the study of SAA in \mathbb{L}^H reduces to determining whether $(Q^{H,\alpha})_{\alpha > 0}$ is entirely asymptotically separable from $(Q^{\frac{\mu\alpha}{\sigma}})_{\alpha > 0}$ or not.²

2.3 Main result

We state now our main result.

Theorem 1. There exists a strong asymptotic arbitrage in the large mixed fractional market \mathbb{L}^H for $\alpha \rightarrow \infty$.

As mentioned we will show that $(Q^{H,\alpha})_{\alpha > 0} \Delta (Q^{\frac{\mu\alpha}{\sigma}})_{\alpha > 0}$.

²In the case when $\mu = 0$, the study of SAA reduces to showing that $(Q^{H,\alpha})_{\alpha > 0} \Delta Q_W$.

3 Proof of Theorem 1

In order to prove Theorem 1, we provide a series of lemmas from which the desired result is obtained as a direct consequence. Before proceeding, we introduce some notations.

Following the lines of [2, 1], we define, for all $n \in \mathbb{N}$, $Y_n : C[0, 1] \rightarrow \mathbb{R}^n$ by:

$$Y_n(\omega) = \left(\omega\left(\frac{1}{n}\right) - \omega(0), \omega\left(\frac{2}{n}\right) - \omega\left(\frac{1}{n}\right), \dots, \omega(1) - \omega\left(\frac{n-1}{n}\right) \right)^T$$

and denote by $Q^{H,\alpha,n}$ and $Q^{\frac{\mu\alpha}{\sigma},n}$ the restrictions of $Q^{H,\alpha}$ and $Q^{\frac{\mu\alpha}{\sigma}}$ to the σ -algebra $\mathcal{F}_n := \sigma(Y_n)$. We fix the Hurst parameter $H \in (3/4, 1)$ and we avoid to mention the dependence on it by setting $Q^{H,\alpha} \equiv Q^\alpha$ and $Q^{H,\alpha,n} \equiv Q^{\alpha,n}$.

We denote by C_n the covariance matrix of the increments of the fractional Brownian motion Z^H :

$$C_n(i, j) := \text{Cov}(Z_{\frac{i}{n}}^H - Z_{\frac{i-1}{n}}^H, Z_{\frac{j}{n}}^H - Z_{\frac{j-1}{n}}^H), \quad 1 \leq i, j \leq n,$$

and by $\lambda_1^n, \dots, \lambda_n^n$ its eigenvalues. Since the matrix C_n is symmetric and positive semi-definite, all the λ_i^n , $1 \leq i \leq n$, are real and nonnegative.

We moreover set

$$\Sigma_0 := \frac{1}{n} I_n + \alpha^2 C_n \quad \text{and} \quad \Sigma_1 := \frac{1}{n} I_n + \frac{1}{n^2} \frac{\mu^2 \alpha^2}{\sigma^2} 1_{n \times n},$$

where I_n is the identity matrix and $1_{n \times n}$ is the $n \times n$ matrix with all coefficients equal to 1. Clearly, the matrices Σ_0 and Σ_1 are positive definite and therefore invertible.

The proof of Theorem 1 strongly relies on the concept of relative entropy (also called sometimes Kullback–Leibler divergence) of the probability measure $Q^{\alpha,n}$ (respectively, Q^α) relative to $Q^{\frac{\mu\alpha}{\sigma},n}$ (respectively, $Q^{\frac{\mu\alpha}{\sigma}}$), denoted by $H(Q^{\alpha,n} | Q^{\frac{\mu\alpha}{\sigma},n})$ (respectively, $H(Q^\alpha | Q^{\frac{\mu\alpha}{\sigma}})$), see [7, Section 6]. We recall the definition of relative entropy and some relevant results in the Appendix A.

Lemma 1. *For each $n \geq 1$, we have*

$$H(Q^{\alpha,n} | Q^{\frac{\mu\alpha}{\sigma},n}) = \frac{1}{2} \left[\text{tr}(\Sigma_1^{-1} \Sigma_0) - n + \frac{\mu^2 \alpha^2}{\sigma^2 n^2} 1_n^T \Sigma_1^{-1} 1_n + \ln \left(\frac{\det(\Sigma_1)}{\det(\Sigma_0)} \right) \right], \quad (3)$$

where $1_n \in \mathbb{R}^n$ is the vector with all coordinates equal to 1, and, for each square matrix A , $\text{tr}(A)$ and $\det(A)$ denote the trace and the determinant of A , respectively.

Proof. Note first that

$$E_{Q^{\alpha,n}}[Y_n Y_n^T] = \Sigma_0 \quad \text{and} \quad E_{Q^{\frac{\mu\alpha}{\sigma},n}}[Y_n Y_n^T] = \Sigma_1.$$

Note also that

$$E_{Q^{\alpha,n}}[Y_n] = 0_n \quad \text{and} \quad E_{Q^{\frac{\mu\alpha}{\sigma},n}}[Y_n] = -\frac{\mu\alpha}{\sigma n} 1_n,$$

where $0_n \in \mathbb{R}^n$ is the vector with all coordinates equal to 0. Since Y_n is a Gaussian vector under the two measures, the result follows using Lemma 7 and performing a straightforward calculation. \square

Using standard properties of the trace and the determinant, it is not difficult to see that

$$\mathrm{tr}(\Sigma_0) = \sum_{i=1}^n \left(\frac{1}{n} + \alpha^2 \lambda_i^n \right) \quad \text{and} \quad \ln(\det(\Sigma_0)) = \sum_{i=1}^n \ln \left(\frac{1}{n} + \alpha^2 \lambda_i^n \right). \quad (4)$$

We set $a_n := \frac{1}{n} \frac{\mu^2 \alpha^2}{\sigma^2}$ and note that $\Sigma_1 = \frac{1}{n}(I_n + a_n 1_{n \times n})$. The next lemma summarizes the properties of the matrix Σ_1 .

Lemma 2. *For each $n > 1$, the eigenvalues of Σ_1 are $1/n$ with multiplicity $n - 1$ and $\frac{1}{n} + a_n$ with multiplicity 1. In particular, we have*

$$\det(\Sigma_1) = \frac{na_n + 1}{n^n}.$$

The inverse of Σ_1 is given by

$$\Sigma_1^{-1} = n \left(I_n - \frac{a_n}{na_n + 1} 1_{n \times n} \right).$$

Proof. Denote $d_n^\lambda := \det(\Sigma_1 - \lambda I_n) = \det\left(\left(\frac{1}{n} - \lambda\right)I_n + \frac{a_n}{n} 1_{n \times n}\right)$. Subtracting the row i from the row $i + 1$, for each $1 \leq i < n$, in the matrix $\Sigma_1 - \lambda I_n$, we see that d_n^λ is equal to the determinant of the matrix

$$\begin{pmatrix} \frac{1}{n} - \lambda + \frac{a_n}{n} & \frac{a_n}{n} & \frac{a_n}{n} & \cdots & \frac{a_n}{n} \\ \lambda - \frac{1}{n} & \frac{1}{n} - \lambda & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \lambda - \frac{1}{n} & \frac{1}{n} - \lambda & 0 \\ 0 & \cdots & 0 & \lambda - \frac{1}{n} & \frac{1}{n} - \lambda \end{pmatrix}. \quad (5)$$

Expanding the determinant by minors with respect to the last column we get

$$d_n^\lambda = \left(\frac{1}{n} - \lambda \right) d_{n-1}^\lambda + \frac{a_n}{n} \left(\frac{1}{n} - \lambda \right)^{n-1}, \quad n > 2.$$

Iterating this identity, we obtain

$$d_n^\lambda = \left(\frac{1}{n} - \lambda \right)^{n-2} d_2^\lambda + (n-2) \frac{a_n}{n} \left(\frac{1}{n} - \lambda \right)^{n-1} = \left(\frac{1}{n} - \lambda \right)^{n-1} \left(\frac{1}{n} - \lambda + a_n \right).$$

The first two statements follow. For the last statement, one can easily check that

$$\Sigma_1 \times n \left(I_n - \frac{a_n}{na_n + 1} 1_{n \times n} \right) = I_n.$$

This shows the desired result. \square

Lemma 3. *For all $n > 1$, we have*

$$\lim_{\alpha \rightarrow \infty} H(Q^{\alpha, n} | Q_{\frac{\mu\alpha}{\sigma}}^n) = \infty.$$

Proof. Our starting point is Lemma 1. Evaluating each term entering (3), we first obtain

$$\begin{aligned} \text{tr}(\Sigma_1^{-1} \Sigma_0) &= n \text{tr}(\Sigma_0) - \frac{na_n}{na_n + 1} \text{tr}(1_{n \times n} \Sigma_0) \\ &= \sum_{i=1}^n (1 + n\alpha^2 \lambda_i^n) - \frac{na_n}{na_n + 1} (1 + \alpha^2 \text{tr}(1_{n \times n} C_n)). \end{aligned} \quad (6)$$

Note that

$$\text{tr}(1_{n \times n} C_n) = \sum_{i,j=1}^n C_n(i, j) = E[(Z_1^H)^2] = 1.$$

Thus, taking $a_n = \frac{1}{n} \frac{\mu^2 \alpha^2}{\sigma^2}$, equation (6) becomes

$$\text{tr}(\Sigma_1^{-1} \Sigma_0) = n + \sum_{i=1}^n n\alpha^2 \lambda_i^n - \frac{\mu^2 \alpha^2}{\mu^2 \alpha^2 + \sigma^2} (1 + \alpha^2). \quad (7)$$

For the third term in (3), using that $1_n^T 1_{n \times n} 1_n = n^2$, one can easily derive that

$$1_n^T \Sigma_1^{-1} 1_n = \frac{n^2}{na_n + 1} = \frac{n^2 \sigma^2}{\mu^2 \alpha^2 + \sigma^2}. \quad (8)$$

For the last term in (3), we use (4) and Lemma 2 to obtain

$$\begin{aligned} \ln\left(\frac{\det(\Sigma_1)}{\det(\Sigma_0)}\right) &= \ln(na_n + 1) - \sum_{i=1}^n \ln(1 + n\alpha^2 \lambda_i^n) \\ &= \ln\left(\frac{\mu^2 \alpha^2 + \sigma^2}{\sigma^2}\right) - \sum_{i=1}^n \ln(1 + n\alpha^2 \lambda_i^n). \end{aligned} \quad (9)$$

Inserting (7), (8) and (9) in (3) yields

$$\begin{aligned} H(Q^{\alpha, n} | Q_{\frac{\mu\alpha}{\sigma}}^n) &= \frac{1}{2} \left[\sum_{i=1}^n (n\alpha^2 \lambda_i^n - \ln(1 + n\alpha^2 \lambda_i^n)) - \frac{\mu^2 \alpha^4}{\mu^2 \alpha^2 + \sigma^2} \right. \\ &\quad \left. + \ln\left(\frac{\mu^2 \alpha^2 + \sigma^2}{\sigma^2}\right) \right]. \end{aligned} \quad (10)$$

Since the trace is similarity-invariant, we deduce that

$$\sum_{i=1}^n \lambda_i^n = \text{tr}(C_n) = \sum_{i=1}^n C_n(i, i) = \frac{1}{n^{2H-1}}.$$

In addition, we have $\ln\left(\frac{\mu^2 \alpha^2 + \sigma^2}{\sigma^2}\right) \geq 0$. Therefore, (10) leads to

$$H(Q^{\alpha, n} | Q_{\frac{\mu\alpha}{\sigma}}^n) \geq \frac{\alpha^2}{2} \left(n^{2-2H} - \frac{\mu^2 \alpha^2}{\mu^2 \alpha^2 + \sigma^2} \right) - \frac{n}{2} \ln(1 + n\alpha^2 \lambda_{\max}^n)$$

$$\begin{aligned}
&\geq \frac{\alpha^2}{2} (n^{2-2H} - 1) - \frac{n}{2} \ln(1 + n\alpha^2 \lambda_{\max}^n) \\
&= \frac{1}{2} \ln\left(\frac{e^{\theta_n \alpha^2}}{(1 + n\alpha^2 \lambda_{\max}^n)^n}\right), \tag{11}
\end{aligned}$$

where $\theta_n := n^{2-2H} - 1 > 0$ and $\lambda_{\max}^n = \max_{i=1\dots n} \lambda_i^n$. The result follows taking the limit when α tends to infinity in the previous expression. \square

Remark 2. If $\mu = 0$, using Lemma 1, the previous result extends directly to the case $n = 1$.

Remark 3. The above proof also gives us the relation between the relative entropy of $Q^{\alpha,n}$ relative to $Q_{\frac{\mu\alpha}{\sigma}}^n$, i.e. $H(Q^{\alpha,n}|Q_{\frac{\mu\alpha}{\sigma}}^n)$, and the relative entropy of $Q^{\alpha,n}$ relative to Q_W^n , i.e. $H(Q^{\alpha,n}|Q_W^n)$. Indeed, using [1, Lemma 5.3] one can deduce from (10) that

$$H(Q^{\alpha,n}|Q_{\frac{\mu\alpha}{\sigma}}^n) = H(Q^{\alpha,n}|Q_W^n) - \frac{1}{2} \frac{\mu^2 \alpha^4}{\mu^2 \alpha^2 + \sigma^2} + \frac{1}{2} \ln\left(\frac{\mu^2 \alpha^2 + \sigma^2}{\sigma^2}\right). \tag{12}$$

Remark 4. We point out that we also have

$$\lim_{\alpha \rightarrow \infty} H(Q^\alpha | Q_{\frac{\mu\alpha}{\sigma}}) = \infty.$$

Indeed, we know from [1, Lemma 5.3] that $\sup_n H(Q^{\alpha,n}|Q_W^n) < \infty$, which directly implies that also $\sup_n H(Q^{\alpha,n}|Q_{\frac{\mu\alpha}{\sigma}}^n) < \infty$. Therefore, applying [7, Lemma 6.3] we obtain

$$H(Q^\alpha | Q_W) = \sup_n H(Q^{\alpha,n}|Q_W^n) \text{ and } H(Q^\alpha | Q_{\frac{\mu\alpha}{\sigma}}) = \sup_n H(Q^{\alpha,n}|Q_{\frac{\mu\alpha}{\sigma}}^n).$$

The statement then follows from the result for the restrictions.

For each $n \geq 1$, we denote the Radon–Nikodym derivative of $Q^{\alpha,n}$ relative to $Q_{\frac{\mu\alpha}{\sigma}}^n$ by

$$L_\alpha^n := \frac{dQ^{\alpha,n}}{dQ_{\frac{\mu\alpha}{\sigma}}^n}.$$

Using [7, Lemma 6.1] (see Lemma 7 in Appendix A), we see that

$$H(Q^{\alpha,n}|Q_{\frac{\mu\alpha}{\sigma}}^n) = E_{Q^{\alpha,n}}[\ln(L_\alpha^n)] = E_{Q_{\frac{\mu\alpha}{\sigma}}^n}[L_\alpha^n \ln(L_\alpha^n)].$$

Moreover, let us recall the notion of $(Q^{\alpha,n})_{\alpha>0}$ -tightness: $(L_\alpha^n)_{\alpha>0}$ is $(Q^{\alpha,n})_{\alpha>0}$ -tight if the following holds:

$$\lim_{N \rightarrow \infty} \limsup_{\alpha \rightarrow \infty} Q^{\alpha,n}(L_\alpha^n > N) = 0.$$

Lemma 4. For each $n > 1$, the family $(L_\alpha^n)_{\alpha>0}$ is not $(Q^{\alpha,n})_{\alpha>0}$ -tight.

Proof. We know, by Lemma 3, that $E_{Q^{\alpha,n}}[\ln(L_\alpha^n)] = H(Q^{\alpha,n} | Q_{\frac{\mu\alpha}{\sigma}}^n)$ tends to infinity when α tends to ∞ . Since the measures $Q^{\alpha,n}$ and $Q_{\frac{\mu\alpha}{\sigma}}^n$ are Gaussian, the result follows as a direct application of the remark on [5, p. 457] which says that tightness is equivalent to the boundedness of the following two families: $E_{Q^{\alpha,n}}[\ln(L_\alpha^n)]$, $\alpha > 0$, and $\text{Var}_{Q^{\alpha,n}}[\ln(L_\alpha^n)]$, $\alpha > 0$. \square

Before we can state and prove the last lemma of this section, we recall now the definition of contiguity of sequences/families of probability measures.

Definition 6. A sequence of probability measures $(P_\ell)_{\ell \geq 1}$ is contiguous with respect to the sequence of probability measures $(Q_\ell)_{\ell \geq 1}$, $(P_\ell)_{\ell \geq 1} \triangleleft (Q_\ell)_{\ell \geq 1}$, if for any sequence $A_\ell \in \mathcal{F}^\ell$: $\lim_{\ell \rightarrow \infty} Q_\ell(A_\ell) = 0 \Rightarrow \lim_{\ell \rightarrow \infty} P_\ell(A_\ell) = 0$. We say that $(P_\ell)_{\ell \geq 1}$ and $(Q_\ell)_{\ell \geq 1}$ are mutually contiguous if $(P_\ell)_{\ell \geq 1} \triangleleft (Q_\ell)_{\ell \geq 1}$ and $(Q_\ell)_{\ell \geq 1} \triangleleft (P_\ell)_{\ell \geq 1}$, in which case we write $(P_\ell)_{\ell \geq 1} \triangleleft \triangleright (Q_\ell)_{\ell \geq 1}$.

These notions extend to families of probability measures $(P^\alpha)_{\alpha > 0}$ and $(Q^\alpha)_{\alpha > 0}$ as follows. We say that $(P^\alpha)_{\alpha > 0}$ is contiguous (resp. mutually contiguous) to $(Q^\alpha)_{\alpha > 0}$ if for every sequence $(\alpha_\ell)_{\ell \geq 1}$ converging to infinity we have $(P^{\alpha_\ell})_{\ell \geq 1} \triangleleft (Q^{\alpha_\ell})_{\ell \geq 1}$ (resp. $(P^{\alpha_\ell})_{\ell \geq 1} \triangleleft \triangleright (Q^{\alpha_\ell})_{\ell \geq 1}$), in which case we write $(P^\alpha)_{\alpha > 0} \triangleleft (Q^\alpha)_{\alpha > 0}$ (resp. $(P^\alpha)_{\alpha > 0} \triangleleft \triangleright (Q^\alpha)_{\alpha > 0}$).

Lemma 5. For each $n > 1$, we have

$$(Q^{\alpha,n})_{\alpha > 0} \triangleleft (Q_{\frac{\mu\alpha}{\sigma}}^n)_{\alpha > 0}.$$

Proof. Since, by Lemma 4, $(L_\alpha^n)_{\alpha > 0}$ is not tight with respect to $(Q^{\alpha,n})_{\alpha > 0}$ we apply [8, Lemma V.1.6] and deduce that, for each $n > 1$, $(Q^{\alpha,n})_{\alpha} \not\triangleleft Q_{\frac{\mu\alpha}{\sigma}}^n$. The dichotomy for sequences of Gaussian measures of [5, Corollary 4] says that two sequences of Gaussian measures on \mathbb{R}^n are either mutually contiguous or entirely separable. So we conclude that, for each $n > 1$, $(Q^{\alpha,n})_{\alpha > 0} \triangleleft (Q_{\frac{\mu\alpha}{\sigma}}^n)_{\alpha > 0}$. \square

Remark 5. From Remark 2, when $\mu = 0$, the same arguments lead to the conclusion that Lemma (5) holds true for $n = 1$.

Proof of Theorem 1. From Proposition 1 (see also [9, Proposition 4]), we know that there is a SAA if and only if $(Q^\alpha)_{\alpha > 0} \triangleleft (Q_{\frac{\mu\alpha}{\sigma}})_{\alpha > 0}$.

Fix $n > 1$. By Lemma 5, there exist a sequence $(\alpha_\ell)_{\ell \geq 1}$ converging to infinity and sets $A_\ell \in \mathcal{F}_n$ such that

$$\lim_{\ell \rightarrow \infty} Q^{\alpha_\ell}(A_\ell) = \lim_{\ell \rightarrow \infty} Q^{\alpha_\ell,n}(A_\ell) = 0$$

and

$$\lim_{\ell \rightarrow \infty} Q_{\frac{\mu\alpha_\ell}{\sigma}}(A_\ell) = \lim_{\ell \rightarrow \infty} Q_{\frac{\mu\alpha_\ell}{\sigma}}^n(A_\ell) = 1.$$

The result follows. \square

4 Interpretation of the results in the restricted markets

Lemma 5 might suggest that, for each $n > 1$ (or following Remark 5, for each $n \geq 1$ if $\mu = 0$), there exists also some kind of asymptotic arbitrage in the large financial market consisting of the restrictions of the α -mixed fractional Black–Scholes models, $\alpha > 0$, to the grid $E_n := \{0, \frac{1}{n}, \dots, \frac{n-1}{n}, 1\}$. However, we will show that this is impossible.

For simplicity, we only consider the case $n = 1$ and $\mu = 0$. We also assume that $S_0^{H,\alpha} = 1$. Thus, for each $\alpha > 0$, the corresponding market is

$$S_t^{H,\alpha} = \exp\left(\sigma\left(Z_t^H + \frac{1}{\alpha}B_t\right) - \frac{\sigma^2}{2\alpha^2}t\right), \quad t = 0, 1. \quad (13)$$

In this case all possible strategies are constants and hence the value process V_1^α takes the following form

$$V_1^\alpha = c_\alpha(S_1^{H,\alpha} - 1),$$

where $c_\alpha \in \mathbb{R}$. Obviously, we cannot hope for admissibility (boundedness from below), see the discussion in the introduction of [10]. But even if we do not require any admissibility here, there is no way to choose a sequence of $\alpha_\ell \rightarrow \infty$ and corresponding value processes V^{α_ℓ} such that the following hold: there exists $\beta > 0$ and $\varepsilon_\ell \rightarrow 0$ with

$$\begin{aligned} (i) \quad & P(V_1^{\alpha_\ell} > \beta) > \beta, \text{ for all } \ell, \\ (ii) \quad & \lim_{\ell \rightarrow \infty} P(V_1^{\alpha_\ell} \geq -\varepsilon_\ell) = 1. \end{aligned} \quad (14)$$

This is not possible since Z_1^H as well as B_1 are independent $N(0, 1)$ and hence are strictly positive as well as strictly negative, with positive P -probability (and here neither letting $\alpha \rightarrow \infty$ nor multiplying $S_1^{H,\alpha} - 1$ by some constants, either positive or negative, will be of any help: whenever there will be a strictly positive part in the limit there will also be a strictly negative part in the limit with a non-disappearing probability). Hence there is no such thing as (14) which, in our discrete time $t = 0, 1$ situation, is the appropriate version of an asymptotic arbitrage.

The reason behind this apparent contradiction is that in contrast to the continuous time large financial market its discrete counterpart is not complete. Under the original measure P (which induces Q^α on $\mathcal{C}[0, 1]$) we have that $Z_1^H \sim N(0, 1)$ and $B_1 \sim N(0, 1)$ and the two random variables are independent. We know that the Wiener measure Q_W is a martingale measure for S^α (understood on $\mathcal{C}[0, 1]$) for all α , hence $Q_W|_{\mathcal{F}_1}$ is an equivalent martingale measure for (13). We will now construct a different martingale measure for the process (13) which is equivalent to P on (Ω, \mathcal{F}) .

Indeed, define a measure \tilde{P} on (Ω, \mathcal{F}) as follows: $\frac{d\tilde{P}}{dP} = g(X)$ where we have $X := \exp(\sigma Z_1^H)$ and $g(x) = e^{-x} \frac{1}{h(x)}$ where $h(x) = \frac{1}{\sqrt{2\pi}\sigma x} \exp(-\frac{1}{2}(\frac{\ln(x)}{\sigma})^2)$ is the density of a lognormal distribution, i.e., the density of the law of X under P . Obviously this measure change has the purpose to make the distribution of X exponential with parameter 1. Recall that the measure $Q^{\alpha,1}$ is considered as a measure on \mathbb{R} (the measure induced by $M_1^{H,\alpha} := \alpha Z_1^H + B_1$).

Lemma 6. *The measure \tilde{P} satisfies:*

1. $\tilde{P} \sim P$.
2. $E_{\tilde{P}}[S_1^{H,\alpha}] = 1 = S_0^{H,\alpha}$, which means that \tilde{P} is a martingale measure for (13), for each α .
3. Let $\tilde{Q}^{\alpha,1}$ be the measure that is induced by $(M_1^{H,\alpha}, \tilde{P})$ on \mathbb{R} , for each $\alpha > 0$. Then $(\tilde{Q}^{\alpha,1})_{\alpha>0} \ll (\tilde{Q}^{\alpha,1})_{\alpha>0}$.

Proof. To prove (1) observe that $g(x) > 0$ for all $0 < x < \infty$ and $0 < X < \infty$ P -a.s. and hence $\frac{d\tilde{P}^\alpha}{dP} = g(X) > 0$ P -a.s. Moreover

$$E_P[g(X)] = \int_0^\infty g(x)h(x)dx = \int_0^\infty e^{-x}dx = 1,$$

and so \tilde{P} is a probability equivalent to P .

For (2) we see that

$$\begin{aligned} E_{\tilde{P}}[S_1^{H,\alpha}] &= E_P \left[g(e^{\sigma Z_1^H}) \exp \left(\sigma \left(Z_1^H + \frac{1}{\alpha} B_1 \right) - \frac{\sigma^2}{2\alpha^2} \right) \right] \\ &= E_P [g(e^{\sigma Z_1^H}) \exp(\sigma Z_1^H)] E_P \left[\exp \left(\frac{\sigma}{\alpha} B_1 - \frac{\sigma^2}{2\alpha^2} \right) \right] \\ &= E_P [g(X)X], \end{aligned}$$

where we used the independence of Z_1^H and B_1 under P . Finally we have that

$$E_P [g(X)X] = \int_0^\infty g(x)xh(x)dx = \int_0^\infty xe^{-x}dx = 1,$$

proving (2).

For (3) note that, for each $A \in \mathcal{B}(\mathbb{R})$, we have $Q^{\alpha,1}(A) = P(M_1^{H,\alpha} \in A)$ and $\tilde{Q}^{\alpha,1}(A) = \tilde{P}(M_1^{H,\alpha} \in A)$. Thus, using that $\tilde{P} \sim P$, we infer, for a family of sets A^α , that $Q^{\alpha,1}(A^\alpha) = P(D^\alpha) \rightarrow 0$, for $\alpha \rightarrow \infty$, if and only if $\tilde{Q}^{\alpha,1}(A^\alpha) = \tilde{P}(D^\alpha) \rightarrow 0$, where $D^\alpha = \{M_1^{H,\alpha} \in A^\alpha\} \in \mathcal{F}_1$. The result follows. \square

In conclusion, Lemma 6 shows that there exists a family of equivalent martingale measures for the model (13) with good properties, in this case with the property of mutual contiguity. And this fact is reflected by the impossibility to find asymptotic arbitrage opportunities for the family of models (13), $\alpha > 0$.

A Relative entropy

In this section, we recall the concept of relative entropy and some equivalent characterization. A more detailed presentation of the topic can be found in [7].

Definition 7. Let Q_1 and Q_2 be probability measures on a measurable space (Ω, \mathcal{F}) and let $P = \{F_i : i = 1, \dots, n\}$ be a finite partition of Ω , i.e. $\Omega = \cup_{i=1}^n F_i$ and F_i are pairwise disjoint. The entropy of the measure Q_1 relative to Q_2 is the quantity

$$H(Q_1|Q_2) = \sup_{\mathcal{P}} \sum_{j=1}^n Q_1(F_j) \ln \left(\frac{Q_1(F_j)}{Q_2(F_j)} \right),$$

where \mathcal{P} is the class of all possible finite partitions P of Ω . In the above formula, we assume that $0 \ln 0 = 0$ and $\ln 0 = -\infty$.

Lemma 7 ([7, Lemma 6.1]). *If a probability measure Q_1 is absolutely continuous w.r.t. another probability measure Q_2 , then the relative entropy $H(Q_1|Q_2)$ is related to the Radon–Nikodym derivative $\varphi = \frac{dQ_1}{dQ_2}$ as follows:*

$$H(Q_1|Q_2) = E_{Q_1}[\ln(\varphi)] = E_{Q_2}[\varphi \ln(\varphi)].$$

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